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An algorithmic construction of low-temperature phase diagrams for layered systems

Maciek Tarnawski

Institute of Physics, Technical University, 50-370 Wroclaw, Poland

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Abstract. We consider phase diagrams for a class of layered systems. Diagrams are studied by means of an inductive argument based on the low-temperature expansions technique. We show that the layered structure of the ground state enhances the emergence of an infinite set of phases. As examples, we consider the ANNNI and the three-state chiral Potts models. We also discuss briefly a new version of the three-state Potts model which exhibits the phase diagram of the devil's staircase type.

1. Introduction

The low-temperature expansions (LTE) technique is often used as the first approximation in studying phase diagrams for spin lattice systems. It has yielded phase diagrams with an infinite number of phases, e.g., for the ANNNI [1] and the three-state chiral Potts [2] models. The important feature of these two models is that at some values of the coupling constants, the set of ground states has a *layered structure*: the lattice can be represented as a sequence of pairwise parallel hyperplanes (layers) with each ground state constant in every hyperplane. Hence the ground state G is identified with a sequence of spin values, corresponding to values of G in consecutive layers. In this paper we study the problem of how the layered structure of ground states enhances the emergence of an infinite set of phases in the phase diagram. The method described here is a generalisation of an algorithm used originally in [3] to study phase diagrams for systems where the rigorous theory of Pirogov and Sinai [4] is applicable. One of the requirements imposed on such systems is that the number of ground states is finite.

For systems with an infinite number of ground states, the LTE phase diagram has to be constructed by means of some inductive pattern. In the analysis of the ANNNI model, Selke and Fisher [1] described such a pattern, hereafter referred to as the SF method. They define an infinite set of *structural variables*, with each ground state corresponding to some particular realisation. The key fact is that the LTE of the pressure can be written as a linear function of structural variables. This representation provides the separation of two basic features of the system. One is the geometry of the set of ground states expressed in terms of structural variables. Another feature is characterised by energetical properties, such as the configurations and energies of excitations in a given local environment. The presssure of the system, which is the maximum of the pressure function, is obtained order by order by means of linear programming procedures.

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In this paper we use a similar idea to study a class of layered systems containing the ANNNI and the three-state chiral Potts models. We show that the layered structure equips the set of ground states with some general properties which enhance the emergence of an infinite set of phases. The final form of the phase diagram depends upon energetical properties which are characteristic for each particular system. The idea behind our analysis is in some way a generalisation of the SF method although technical details are different. We also begin by representing the pressure in terms of structural variables, but with another arrangement of terms. Furthermore, each ground state is identified by a sequence of spin values rather than by a realisation of structural variables. Inductive steps of both methods can be compared as follows.

(i) In the preceding step, one defines a set of phases coexisting at some value of the coupling constants. In the SF method, this set is given by the condition of consistency (in the sense of the consistency lemma) with two particular phases. Propositions II and III identify consistent phases. In our method, each element of the set is defined by a sequence written as a combination of two particular sequences (ground states). We show that phases identified in the SF method by propositions II and III can be reduced to this form.

(ii) In the small neighbourhood of the coexistence point, the pressure function is reduced (in the leading order). In the SF method the reduced form consists of only one structural variable. The coefficient of this variable is then calculated in the leading order. In our method, the reduced form is an affine functional with the constant term consisting of just one structual variable. The reduction is done by means of proposition I. The search for the corresponding coefficient, which is admittedly the hardest part of the analysis, has to be conducted for each system individually.

(iii) The reduced pressure function is maximised and this process defines also the set of phases to be considered in the next step. The SF method uses standard linear programming procedures. In our method, we apply the construction of the phase diagram for a set of affine functionals (lemma in appendix 1). The result is described by theorem 2. Depending upon the sign of the coefficient in (ii), the inductive step either shows the existence of a new phase and reconstructs conditions for the next step, or no new phases ever appear in the neighbourhood of the coexistence point.

The paper is laid out as follows. In § 2 we define a layered system, and construct the representation of the LTE of the pressure in terms of structural variables. In § 3, the general idea is presented of the LTE phase diagram construction. Section 4 contains the description of an inductive step. Throughout §§ 2–4, the ANNNI model is used as the example. In § 5 we briefly describe two other examples. One is the three-state Potts model [2] studied orginally by the SF method. The other example is a new version of the three-state Potts model which has the phase diagram of the devil's staircase type. In the appendices we present the phase diagram construction for affine functionals and give the proof of theorem 2.

2. The low-temperature expansions for layered systems

We consider a system defined on the $\mathbb{Z}^3$ lattice, with a finite configuration set in any lattice point. The system is described by a classical finite range Hamiltonian H_0. A configuration G is called a *ground state* of H_0 if, for any configuration X which differs from G in a finite set of lattice points (a support of X),

$$E(X) \equiv H_0(X) - H_0(G) \geqslant 0.$$

The configuration X is called an *excitation* of G. We assume that H_0 satisfies the sufficient condition for the existence of the low-temperature expansion (LTE) of the pressure in the G boundary conditions: If $\{X_n\}$ is a sequence of excitations with support increasing to infinity, then $E(X_n)$ diverges [5].

Let us suppose now that the set of ground states has the following structure. There exists a lattice line λ such that in any *layer* Q (i.e. hyperplane perpendicular to λ) each ground state is a constant configuration. A system with such a property is called a *layered system*. Thus any ground state is described by a sequence of spin values $\ldots G_1, G_2 \ldots$, with G_i being the value of G in the ith layer. This sequence will be also denoted as G.

Next we consider a collection of Hamiltonians of the form

$$H_0 + xL \qquad x \text{ in a neighbourhood of zero} \tag{2.1}$$

where L is a finite range Hamiltonian and we assume that it partially lifts the degeneracy of the ground state. This means that for small values of x, the set of ground state of (2.1) is smaller than the set of ground states for the non-pertubed Hamiltonian.

The simplest way of constructing a layered system is to consider a Hamiltonian H_0 with an anisotropic interaction, which is ferromagnetic in each layer and competitive in its interlayer part. Examples of such systems are the ANNNI and the three-state chiral Potts models. The generalisation is given by the following condition.

Condition 1. Let G be any ground state of H_0, and X be an excitation of G. If X' is another excitation obtained from X by exciting a new layer, then $E(X') - E(X) > 0$. (A layer is excited if it contains points of support.)

Example: the ANNNI model. This model is described by the Hamiltonian ($s_i = +, -$):

$$H(J_2) = -J_0 \sum_{nn}^{\perp} s_i s_j - J_1 \sum_{nn}^{\parallel} s_i s_{i+1} + J_2 \sum_{nnn}^{\parallel} s_i s_{i+2}.$$

Here $J_i > 0$. The first sum denotes the nearest-neighbour ferromagnetic interaction in planes parallel to the yz plane, while the second and third sums correspond to the respectively nearest- and next-nearest-neighbour interaction along the x axis. The Hamiltonian H_0 is $H(J_2 = \frac{1}{2}J_1)$. Each ground state of H_0 is constant in any layer (a plane parallel to the yz plane). The spin sign in different layers is arbitrary with the restriction that at least two neighbouring layers must have the same sign (i.e. the configuration $+-+$ is not allowed).

The perturbation of H_0 is described by the deviation of the NNN coupling constant J_2 from the value $\frac{1}{2}J_1$: $J_2 = \frac{1}{2}J_1 + x$.

Our main goal is to study for layered systems the phase diagram resulting from the LTE technique. In this technique, each phase corresponds to a periodic ground state G and is described by the LTE of the pressure in G boundary conditions:

$$-xe_G + \sum_{i=1}^{\infty} n_i^G(x)\, \mathrm{e}^{-\beta E_i} \qquad x \in \mathbb{R} \tag{2.2}$$

with $\beta = 1/T$ incorporated into x. Here e_G is the average energy per lattice site of the Hamiltonian L in the ground state G, and E_i corresponds to an excitation energy $E(X)$ for some excitation X of G. The coefficient $n_i^G(x)$ is the sum of terms $\exp[x(L(X) - L(G))]$ for all excitations X (modulo translations) with energy E_i (cf also [6]).

Since each ground state G is identified with a sequence of spin values, the expansion coefficient n_i^G can be written in terms of finite subsequences. The heuristic argument for this fact can be found in [1]. Let A be a finite sequence of spin values. In N consecutive layers A appears $N_A(G) = l_A(G)N$ times (modulo boundary terms) as a subsequence of G. The number $l_A(G)$ is the density of A in G. Following [1], it will be called a *structural variable.* Since in the LTE technique we restrict our attention to periodic ground states only, $l_A(G)$ is well defined.

Structural variables satisfy relations ($|B|$ is the length of B):

$$l_A(G) = \sum_{|B|=b} l_{AB}(G) = \sum_{|B|=b} l_{BA}(G). \tag{2.3}$$

In particular, $\Sigma_\sigma l_\sigma(G) = 1$, with summation over all spin values σ.

With the help of structural variables, the coefficient n_i^G can be written in the form

$$n_i^G(x) = \sum_{|A|\leq r_i} a_{i,A}(x) l_A(G). \tag{2.4}$$

The restriction on $|A|$ comes from the finiteness of the range of H_0 and condition 1. Inserting (2.4) into (2.2), one obtains the representation of the LTE by structural variables. This representation is different from the one used in [1], where all terms corresponding to a given structural variable l_A have been collected into one coefficient a_A. Since the series (2.2) is not necessarily convergent, a_A may be divergent. Therefore we will use the arrangement of the series in which all terms corresponding to a given excitation energy E_i are collected into n_i^G.

The coefficient e_G of the linear term in (2.2) also can be expressed by structural variables, but the exact form depends upon the perturbation L and cannot be generalised.

In the ANNNI model, we consider sequences (k) defined as a band of k spins with the same sign bordered by spins of the opposite sign. The corresponding structural variable is denoted by $l_k(G)$. The sequence (k) can be realised in two ways, differing by the sign of the first spin. In examples, we will choose realisation with the first + spin.

3. The phase diagram for layered systems: general construction

The infinite series appearing in (2.2) is potentially divergent, and nothing is in general known about its behaviour (for example, if it is asymptotic). Therefore it is truncated at an arbitrary finite order, say N. The phase diagram in this order is obtained by maximising, for each value of the parameter x separately, the expression:

$$-xe_G + \sum_{i=1}^{N} n_i^G(x)\, \mathrm{e}^{-\beta E_i}. \tag{3.1}$$

There are two possible outcomes of maximisation. First (3.1) can be maximised by a single ground state G and in this case x falls into the region occupied by the phase G. Alternatively, several ground states maximise (3.1) simultaneously. In this case $x = x(\beta)$ is the line of coexistence of corresponding phases and also the borderline between two regions. The borderline is potentially unstable, i.e. the inclusion of higher-order terms in (3.1) may result in the appearance of new phases at its locus. The higher-order analysis is conducted in the following way. One argues that new terms introduce corrections $o[\exp(-\beta E_N)]$. Hence it is enough to study only a small neighbourhood of the borderline $x(\beta)$, within which one can restrict attention to

ground states coexisting at $x(\beta)$ at order N. We should remark here that in view of the potential divergence of (2.2) the meaning of a 'small neighbourhood' is not clear. This problem is embedded into the wider question about the interpretation of the LTE phase diagram. Except for special systems with a finite number of ground states (cf [5] where the phase diagram is shown to be asymptotic to the rigorous one) this problem has not been solved so far.

Let us now describe briefly the method for studying the maximisation of (3.1). In the zero order (corresponding to zero temperature) (3.1) consists only of the linear term $-xe_G$. Hence the region of negative x is occupied by ground states with maximal e_G, while positive values of x fall into the region occupied by ground states with minimal e_G. All other ground states coexist at zero. Hence the analysis can be restricted in the first order to a small neighbourhood of the origin.

Suppose now that in the Nth step we consider a borderline x_{N-1} defined in order $N-1$ with several coexisting phases. In a small neighbourhood of x_{N-1} of radius $o[\exp(-\beta E_{N-1})]$, (3.1) assumes the form:

$$-xe_G + \sum_{i=1}^{N} n_i^G(x_{N-1})\, e^{-\beta E_i} + o(e^{-\beta E_N}). \tag{3.2}$$

The correction term, which can be disregarded in order N, arises from higher power terms of the Taylor expansion of n_i^G around x_{N-1}. Hence (3.2) is an affine functional and maximisation of (3.1) reduces to the problem of finding the phase diagram for a set of affine functionals. This problem is discussed in appendix 2.

The zero- and first-order analysis of the ANNNI model gives the same results as obtained by the SF method. In zero order the region of negative x is occupied by the ferromagnetic ground state and the region of positive x by the class of ground states denoted by $\langle 2\rangle$ (the periodic repetition of the sequence (2)). In first order the new phase appears, defined by the periodic repetition of the sequence (3) and denoted as $\langle 3\rangle$. The next-order analysis of the boundary between $\langle 3\rangle$ and the ferro phase shows that this boundary is stable. We will concentrate on the boundary $\langle 3\rangle$-$\langle 2\rangle$. Phases coexisting there satisfy the condition: $l_k(G) = 0$ if $k \geq 4$ and hence they have the form:

$$\langle 2^{p_1}3^{q_1} \ldots 2^{p_s}3^{q_s}\rangle \qquad p_i, q_i \geq 1 \text{ if } s > 1. \tag{3.3}$$

It is not hard to see that for ground states (3.3)

$$e_G = -l_3(G) - 2l_2(G) = -\tfrac{1}{3} - \tfrac{4}{3}l_2(G). \tag{3.4}$$

4. The inductive step

Let us suppose that the set of ground states of the Hamiltonian H_0 is infinite. The basic question is then whether the infinite set of phases appears in the phase diagram. The way to study this problem is to find an inductive pattern governing the behaviour of the phase diagram in increasing orders. In this section we show that the layered structure of ground states enhances the emergence of such a pattern. The appearance of the infinite set of phases depends upon specific properties of the system manifesting themselves in the first orders of the analysis (constituting the starting point of induction) and in the form of coefficients associated with the structural variables.

The inductive step consists of four parts:

(i) defining a subset of ground states, which is given in the preceding order as a collection of ground states coexisting at some borderline,

(ii) determining the order N in which the series (2.2) is truncated,

(iii) reducing the form of the coefficient n_N^G,

(iv) constructing the phase diagram, based on the corresponding construction for affine functionals.

4.1. The choice of a subset of ground states

Let r be the range of the interlayer part of the Hamiltonian H_0+xL: a spin in the ith layer can interact at most with spins in layers $i\pm k$, $k\leqslant r$. Furthermore let A, B be sequences of spin values. We consider a set of ground states in the form:

$$\langle A^{p_1}(A^uB)^{q_1}A^{p_2}\ldots A^{p_s}(A^uB)^{q_s}\rangle \qquad p_i,\ q_i\geqslant 1 \text{ if } s>1 \tag{4.1}$$

where $A^p=AA\ldots A$ (p times) and $\langle A\rangle$ denotes an infinite sequence obtained by periodic repetition of A.

Sequences A, B and the number u satisfy the following condition.

Condition 2. $|A|\geqslant r$, and if $u=0$ then $|B|\geqslant r$.

We assume furthermore that there exist numbers a_1, a_2, b_1, b_2 such that

$$l_{A^{u+1}}(G)=a_1e_G+b_1 \qquad l_{A^uB}(G)=a_2e_G+b_2. \tag{4.2}$$

We will show that the form (4.1) and the conditions (4.2) are reconstructed at the end of the inductive step, with new definitions of A and B.

In the ANNNI model, we will consider two generic cases.

(a) $A=(2)$ (two-layer band), $B=(3)$ (three-layer band), $u=j$. Then (4.1) and (4.2) become:

$$\langle 2^{p_1}(2^j3)^{q_1}\ldots\rangle \tag{4.3}$$

and

$$l_{2^{j+1}}(G)=a_1l_2(G)+b_1 \qquad l_{2^j3}(G)=a_2l_2(G)+b_2. \tag{4.4}$$

(b) $A=(32^{j-1})$, $B=(2)$, $u=1$. Then (4.1) and (4.2) become:

$$\langle(32^{j-1})^{p_1}(32^j)^{q_1}\ldots\rangle \tag{4.5}$$

and

$$l_{32^{j-1}32^{j-1}}(G)=a_1'l_2(G)+b_1' \qquad l_{32^j}(G)=a_2'l_2(G)+b_2'. \tag{4.6}$$

We remark that in the SF method, the choice of a subset of ground states is a special case of (4.1). Namely, one considers there ground states consistent with a pair of vertices of two special forms, as follows.

(i) Vertices $\langle 2\rangle$ and $\langle 2^j3\rangle$, The inductive application of consistency with vertices $\langle 2\rangle$ and $\langle 2^k3\rangle$, $k\leqslant j$, gives conditions $l_{32^{k-1}3}(G)=0$, and hence G has the form (4.3). Moreover, by proposition II of [1], (4.4) is also reproduced.

(ii) Vertices $\langle 2^{j-1}3\rangle$ and $\langle 2^j3\rangle$. Consistent ground states satisfy conditions: $l_{32^{k-1}3}(G)=0$, $k\leqslant j-1$, and $l_{2^{k+1}}(G)=0$. Hence ground states have the form (4.5). Condition (4.6) is the consequence of proposition III of [1].

4.2. The order of truncation

The choice of the order N in which (2.2) is truncated is connected with the following observation. Let x_N be the borderline in order N between ground states $\langle A\rangle$ and $\langle A^uB\rangle$. Suppose that for all ground states (4.1)

$$n_i^G(x_N) = c_i e_G + d_i \qquad i \leqslant N. \tag{4.7}$$

Then (2.3) has the form

$$(x_N - x)e_G + \sum_{i=1}^{N} d_i \, e^{-\beta E_i} + o(e^{-\beta E_\wedge}). \tag{4.8}$$

It is not hard to see that e_G lies between $e_{\langle A\rangle}$ and $e_{\langle A^uB\rangle}$. Then the maximisation of (4.8) results in a trivial modification of the zero-order phase diagram: all ground states coexist at x_N and each of two regions defined by x_N is occupied by one of the phases $\langle A\rangle$ or $\langle A^uB\rangle$. To obtain a non-trivial phase diagram, we have to find the lowest order in which (4.7) does not hold. Such an order is determined by the geometry of sequences A and A^uB. Before going into details, let us introduce the following definitions.

A sequence M is an *extension* of a sequence M' if M' is a subsequence of M. M is a *proper* extension of M' if for all ground states (4.1), $l_M(G) = l_{M'}(G)$. This can happen in two cases:

(i) M' is not a subsequence of any form (4.1), and

(ii) M' is a subsequence of one of forms (4.1) and M is the only extension of M' which also is a subsequence of the same form.

Now we define the *common core* C to be the longest subsequence of both $\langle A\rangle$ and $\langle A^uB\rangle$. C is the proper extension of A^u, but it does not have a proper extension itself. There exist spin values μ, $\bar{\mu}$, $\nu \neq \mu$, $\bar{\nu} \neq \bar{\mu}$ such that the following hold.

(i) $C\bar{\mu}$, μC have common proper extensions with A^{u+1}.

(ii) $C\bar{\nu}$ has a common proper extension with A^uB, and νC with BA^u.

(iii) $\mu C\bar{\mu}$ has a common proper extension with A^{u+2}.

(iv) $\nu C\nu$ has a common proper extension with BA^uB.

(v) $\mu C\bar{\nu}$ has a proper common extension with $A^{u+1}B$, and $\nu C\bar{\mu}$ with BA^{u+1}.

In the ANNNI model, case (a), the common core is $(++)2^j(++)$ (j odd) or $(++)2^j(--)$ (j even). For j odd, $(+)C(+) = 32^j3$, $(+)C(-)$ is the proper extension of 32^{j+1}, etc. In case (b), the common core is $(++)2^{j-1}32^{j-1}(++)$ (j odd), or $(++)2^{j-1}32^{j-1}(--)$ (j even). Here $(+)C(+) = 32^{j-1}32^{j-1}3$ which extends to A^3, $(+)C(-)$ is the proper extension of $32^{j-1}32^j = A^2B$ etc.

Recall that the expansion coefficient $n_i^G(x)$ consists of structural variables l_A with $|A| \leqslant r_i$. It turns out that if $r_i \leqslant |C| + 1$, then the corresponding coefficient has the form (4.8).

Proposition 1. Suppose that (4.2) holds. If a sequence M is not an extension of $\alpha C\bar{\beta}$ ($\alpha, \beta = \mu, \nu$) then there exist constants a_M, b_M such that for any ground state (4.1)

$$l_M(G) = a_M e_G + b_M. \tag{4.9}$$

Proof. We define two auxiliary sequences. C_A is the common extension of $C\bar{\mu}$ to the right and μC to the left. It consists of the $C\bar{\mu}$ part, the μC part (possibly overlapping) and the intermediate part which is present only if $C\bar{\mu}$ and μC do not overlap. For example, in the latter case

$$C_A = C\bar{\mu}D\mu C \qquad D = \text{intermediate part.}$$

C_A is the proper extension of A^{u+1}. Similarly we define C_B to be the shortest common extension of $C\bar{\nu}$ and νC. C_B is the proper extension of A^uBA^u. Any sequence (4.1) can be covered by translates of C_A and C_B in such a way that any subsequence C belongs to two translates.

Now let M be a subsequence of one of the forms (4.1). One can find a translate of C_A or C_B such that the left terminal point of M lies in this translate and to the left of its αC part. Let the relevant translate be of C_A. There are three possibilities.

(i) M is contained in C. If d measures the occurrence of M in C, then

$$l_M(G) = dl_C(G) = dl_{A^u}(G) = d(2l_{A^{u+1}}(G) + l_{A^uB}(G)).$$

(ii) M is contained in C_A but not in C. Then

$$l_M(G) = \text{constant} \times l_{C_A}(G) = \text{constant} \times l_{A^{u+1}}(G).$$

(iii) M is not contained in C_A, i.e. the right terminal point of M lies to the right of the $\bar{\mu}C$ part. Hence M contains $\mu C\bar{\mu}$ or $\mu C\bar{\nu}$. A similar argument holds if the left terminal point of M lies in C_B.

4.3. The reduction of the coefficient n_N^G

Let N be the lowest order in which the expansion coefficient contains a structural variable l_M not fulfilling (4.9). By proposition 1, M is an extension of $\alpha C\bar{\beta}$ with $\alpha, \beta = \mu, \nu$. We will show that conditions 1 and 2 force M to be the proper extension.

Suppose that M is an extension of $\mu C\bar{\mu}$ and it is not a proper extension. Let C' be the common core of $\langle A\rangle$ and $\langle A^{u+1}B\rangle$. If C is written as $C_1A^uC_2$, then it is easy to see that $C' = C_1A^{u+1}C_2$. Moreover C' is the proper extension of $C\bar{\mu}$ (and also of μC). Since M is not a proper extension, it has to contain $\mu C'\bar{\beta}$, so

$$|M| \geqslant |\mu C'\bar{\beta}| = |\mu C\bar{\mu}| + |A| \geqslant |\mu C\bar{\mu}| + r.$$

If X is an excitation of M, then by removing the rightmost (and possibly some other) excited layers, we obtain an excitation Y of a *proper extension* (possibly trivial) of $\mu C\bar{\mu}$. By condition 1, $E(X) > E(Y)$. Hence l_M does not enter into the Nth coefficient. A similar argument holds if M is an extension of $\nu C\bar{\mu}$ and for extensions of $\alpha C\bar{\nu}$ then C' is replaced by the common core of $\langle A^uB\rangle$ and $\langle A^{u+1}B\rangle$. Thus we have shown that

$$n_N^G(x) = \sum_{\alpha,\beta=\mu,\nu} l_{\alpha C\bar{\beta}}(G)a_{N,\alpha\bar{\beta}}(x) + c'_N e_G + d'_N. \tag{4.10}$$

Here $a_{N,\alpha\bar{\beta}}(x)$ is the sum of coefficients $a_{N,M}(x)$ for proper extensions M of $\alpha C\bar{\beta}$. With the help of structural relations (2.3), all $l_\alpha C_{\bar{\beta}}$ can be expressed by one of them, say $l_{\mu C\bar{\mu}}$. All other variables entering structural relations satisfy (4.9). If we define

$$a_N(x) = a_{N,\mu\bar{\mu}}(x) + a_{N,\nu\bar{\nu}}(x) - a_{N,\mu\bar{\nu}}(x) - a_{N\nu\bar{\mu}}(x)$$

then (4.10) takes the form

$$n_N^G(x) = a_n(x)l_{\mu C\bar{\mu}}(G) + c_N e_G + d_N. \tag{4.11}$$

We remark that the reduced form (4.11) has been derived here from conditions 1 and 2. The following part of the inductive step assumes only (4.11) so it is valid even if conditions 1 and 2 do not hold.

4.4. The phase diagram for ground states (4.1)

Let x_{N-1} and x_N be boundaries between phases $\langle A\rangle$ and $\langle A^uB\rangle$ defined in orders $N-1$ and N, respectively. All ground states (4.1) coexist at x_{N-1}. In the small neighbourhood of x_{N-1} of radius $o[\exp(-\beta E_{N-1})]$ (4.11) can be evaluated at x_N, the correction term of order higher than N being dropped. The constant term is also dropped, and the term proportional to e_G enters into the definition of x_N. The phase diagram for ground states (4.1) is determined by maximising the affine functional:

$$-(x-x_N)e_G + a_N(x_N)l_{\mu C\bar{\mu}}(G)\,\mathrm{e}^{-\beta E_N} \tag{4.12}$$

The outcome of the maximisation is described by the following theorem.

Theorem 2.

(1) If $a_N(x_N)>0$, then the boundary x_N is stable. No phases of the form (4.1) appear in the phase diagram, except for $\langle A\rangle$ and $\langle A^uB\rangle$.

(2) If $a_N(x_N)<0$, then the boundary x_N is unstable. The new phase $\langle A^{u+1}B\rangle$ appears between $\langle A\rangle$ and $\langle A^uB\rangle$. The width of the region occupied by this phase is in the leading order $d\exp(-\beta E_N)$ with

$$d=\frac{|a_N|}{|A^{u+1}B|}\left|\frac{1}{e_{\langle A^{u+1}B\rangle}-e_{\langle A\rangle}}-\frac{1}{e_{\langle A^{u+1}B\rangle}-e_{\langle A^uB\rangle}}\right|$$

(a) Phases which coexist at the boundary x'_N between $\langle A\rangle$ and $\langle A^{u+1}B\rangle$ are described by the condition: $l_{BA^uB}(G)=0$, i.e.

$$G=\langle A^{p_1}(A^{u+1}B)^{q_1}\ldots\rangle. \tag{4.13}$$

Moreover (4.2) is reproduced, i.e.

$$l_{A^{u+2}}(G)=a'_1\cdot e_G+b'_1 \qquad l_{A^{u+1}B}(G)=a'_2e_G+b'_2.$$

(b) Phases which coexist at the boundary x''_N between $\langle A^uB\rangle$ and $\langle A^{u+1}B\rangle$ are described by the condition: $l_{A^{u+2}}(G)=0$, i.e.

$$G=\langle (BA^u)^{p_1}(BA^{u+1})^{q_1}\ldots\rangle. \tag{4.14}$$

Moreover (4.2) is also reproduced, i.e.

$$l_{BA^uBA^u}(G)=a''_1e_G+b''_1 \qquad l_{A^{u+1}B}(G)=a''_2e_G+b''_2.$$

The proof of theorem 2 is contained in appendix 2. We remark that in the case $a_N(x_N)=0$, one has to consider the next order $N+1$. As long as the $(N+1)$th coefficient can be written in the form (4.11), theorem 2 can be applied. If (4.11) does not hold, the inductive step breaks down.

Theorem 2 closes the inductive step of our method. It reconstructs the form (4.1) of the set of ground states and conditions (4.2). Obviously some form of inductive argument is necessary to determine the sign of the coefficient a_N. Condition 2 defines the lowest order in which induction starts. Lower-order analysis is done by the direct application of the construction for affine functionals, described in appendix 1.

In the ANNNI model, ground states coexisting at the boundary $\langle 2\rangle$-$\langle 3\rangle$ have the form (4.1). Hence the induction starts already in the first order. Condition 2 is satisfied, with (3.4) and $2l_2(G)+3l_3(G)=1$. In the inductive step, case (a), we have

$$a_{N_j}(x_{N_j})=-a_{2^j3}(0)-a_{32^j}(0)=2(j+2).$$

The right-hand side has been obtained from the coefficient b_{2^j3} of [1] by taking the lowest-order term. The new phase $\langle 2^{j+1}3\rangle$ appears between $\langle 2^j3\rangle$ and $\langle 2\rangle$. At the boundary $\langle 2^{j+1}3\rangle$-$\langle 2\rangle$, coexisting ground states satisfy the condition: $l_{32^j3}(G)=0$, i.e. it is the repetition of case (a). At the boundary $\langle 2^{j+1}\rangle$-$\langle 2^j3\rangle$, coexisting ground states satisfy the condition: $l_{2^{j+1}}(G)=0$, hence we are the the case (b). Here

$$a_{N_{ll}}(x_{N_{ll}}) = -a_{2^j32^{j+1}3}(0) - a_{32^{j+1}32^j}(0) = 2.$$

Hence the boundary $\langle 2^j3\rangle - \langle 2^{j+1}3\rangle$ is stable. In this way the inductive argument shows the existence of an infinite sequence of phases found in [1]: $\langle 3\rangle \to \langle 23\rangle \to \langle 223\rangle \to \ldots \to \langle 2^j3\rangle \to \ldots \to \langle 2\rangle$.

The width of the region occupied by $\langle 2^j3\rangle$ behaves in the leading order as $c_j \exp[-\beta(j+1)J_0]$ (cf [1]), hence it increases exponentially as the temperature increases. In [7] it is argued that this exponential growth is modified at somewhat higher temperatures so that the phase $\langle 2^j3\rangle$ disappears above some temperature $T_j \to 0$ as $j \to \infty$. Hence at finite temperatures, only a finite number of phases exists. The argument is essentially based on combining two terms of the LTE. In our notation, it corresponds to combining coefficients of order N and $N+1$. It can be shown that the latter one has the form (4.11), hence the overall coefficient preceding $l_{\mu C\bar{\mu}}(G)$ in (4.12) is

$$a_N(x_N)\,\mathrm{e}^{-\beta E_N} + a_{N_{N-1}}(x_{N_{N+1}})\,\mathrm{e}^{-\beta E_{N+1}}.$$

The interplay between these two terms may lead to vanishing of $\langle 2^j3\rangle$ at some finite temperature. However, we believe that because of the potential divergence of the LTE, such a combination of terms *is not justified.* The LTE technique may be applied only in the temperature region where higher-order terms do not influence the lower-order phase diagram in the sense described above. In systems with an infinite number of ground states this region may be reduced to zero temperature only. Thus we arrive again at the problem of the LTE phase diagram interpretation, already mentioned at the beginning of this section.

5. Other examples

5.1. The three-state chiral Potts model

This model has been studied in [2] by the SF method from which the form of expansion coefficients has been adopted. We use a modified Hamiltonian which is equivalent to the original one. Spin values lie in $\mathbb{Z}_3$ (a group $\{0, 1, 2\}$ with addition modulo 3). The system is described by the Hamiltonian:

$$H(\kappa) = -\sum_a \sum_{i=0}^{2} (\kappa P^i_a P^{i+1}_{a+e_1} + P^i_a P^i_{a+e_1}) - J_0 \sum_a \sum_{i=0}^{2} \sum_{k\neq 1} P^i_a P^i_{a+e_k}.$$

Here e_k denotes the base vectors of the cubic lattice, and P^i_a is the projection on the spin value i at the lattice site a. The second term is the NN ferromagnetic interaction between spins lying in the same plane parallel to the yz plane. It forces each ground state to be constant in every such plane (layer). The first term describes the 'chiral' interaction. At $\kappa = 1$, it forces ground state values in the ith and $(i+1)$th layer to satisfy one of the conditions: $G_{i+1} = G_i$; $G_{i+1} = G_i + 1$. Instead of using absolute values, we may provide information only about the jump in spin with passing from the ith to

$(i+1)$th layer. Hence each ground state of $H_0 = H(\kappa = 1)$ is described by a sequence of zeros and ones. One should provide information also about spin value at the origin, but this distinction leads to symmetrical ground states and will be ignored. The perturbation of H_0 is given by a small variation of the coupling constant κ around $1: \kappa = 1 + x$. It is not hard to see that up to the constant, $e_G = -l_0(G)$. Hence the region of negative x is occupied by ground states for which $l_0(G) = 0$. This condition defines a class of ground states denoted by $\langle 1 \rangle$ and consisting of $\langle 012 \rangle$, $\langle 120 \rangle$, $\langle 201 \rangle$. Positive values of x fall into the region occupied by the class $\langle 0 \rangle (l_0(G) = 1)$, which consists of $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle 2 \rangle$. All other ground states coexist at zero.

In the first order, $n_1^G(0) = l_{01}(G) + l_{10}(G)$. Using the phase diagram construction for affine functionals $(-e_G, n_1^G(0))$, one shows that the new phase $\langle 01 \rangle$ (denoted by $\langle 2 \rangle$ in [2]) appears between $\langle 0 \rangle$ and $\langle 1 \rangle$. The phase diagram is symmetric with respect to the line $x = 0$, so we restrict our attention to the boundary $\langle 01 \rangle$-$\langle 0 \rangle$. Phases coexisting there satisfy the condition $l_{11}(G) = 0$ so they are of the form (4.1) with $A = (0)$ and $B = (1)$. Evidently conditions 1 and 2 are satisfied so order one provides the starting point for induction. In the inductive step, we consider two collections of ground states.

(i) $A = (0)$, $B = (1)$, $u = j - 1$. The common core is $C = 0^{j-1}$. From [2] we have that in the relevant order, $a_N(x_N) = -a_{\nu_i} < 0$.

(ii) $A = (10^{j-1})$, $B = (0)$, $u = 1$. The common core is $C = 0^{j-1}10^{j-1}$. From [2], $a_N(x_N) = -a_{\nu_{ij}} > 0$.

Hence the sequence of phases looks as follows. Between phases $\langle 0 \rangle$ and $\langle 01 \rangle$, the phase $\langle 001 \rangle$ appears. The bounday $\langle 01 \rangle$-$\langle 001 \rangle$ is stable, while at the boundary $\langle 0 \rangle$-$\langle 001 \rangle$, the new phase $\langle 0^3 1 \rangle$ shows up in some higher order. In general, the boundary between $\langle 0^{j-1}1 \rangle$ and $\langle 0^j 1 \rangle$ remains stable, while at the boundary $\langle 0^j 1 \rangle$-$\langle 0 \rangle$, the phase $\langle 0^{j+1}1 \rangle$ appears. This is identical as the result of [2].

5.2. The three-state Potts model with NNN interaction

In order to show that our method gives more than just the description of well known systems, we have studied a new version of the three-state Potts model. For lack of space we will not present the full argument here. It will be described at a later time, along with calculations of the LTE coefficients.

The model is defined on the $\mathbb{Z}^3$ lattice, with spin values in $\mathbb{Z}_3$. We define the Hamiltonian

$$H(x) = -(1+x) \sum_a \sum_{i=0}^{2} \sum_{k=1}^{3} P_a^i P_{a+e_k}^{i+1} - \tfrac{1}{2} \sum_a \sum_{i=0}^{2} \sum_{k<l} P_a^i P_{a \pm e_k \pm e_l}^i$$

(cf § 5.1 for notation). The second term is ferromagnetic and has a double meaning. One part of it forces ground states to be constant in each plane perpendicular to the $(1, 1, 1)$ axis (layer). Another part, combined with the first term, gives at $x = 0$ the competitive interlayer interaction which results in an infinite set of ground states. Condition 1 is satisfied by $H_0 = H(0)$. We introduce the following notation. A triple layer $(i-1, i, i+1)$ is denoted by 'o' if $G_{i+1} = G_i + 1$ and $G_i = G_{i-1} + 1$. The triple is denoted by 'α' if $G_{i+1} = G_{i-1}$ with $G_i \neq G_{i+1}$. One can show that each ground state of H_0 corresponds to a sequence of o and α^2 symbols. We will write α^{2k} to denote a sequence of $2k+2$ layers of the type: 0101 ... 01, and o^k corresponds to the sequence: 01201201 ... with $k+2$ elements. In this notation, $\langle \alpha^2 o \alpha^4 o \rangle$ is exemplified by the periodic repetition of the sequence: 010121212020101012120202 (two other representations are obtained if one adds to the above sequence uniformly 1 or 2 modulo 3).

The Hamiltonian H_0 is perturbed by a variation of the nearest-neighbour coupling constant around one. Up to a constant term, $e_G = -l_o(G)$. Hence the region of positive x is occupied by the class $\langle o\rangle$; $\langle 012\rangle$, $\langle 120\rangle$, $\langle 210\rangle$. Negative values of x fall into the region occupied by the class $\langle \alpha^2\rangle$, consisting of $\langle 01\rangle$, $\langle 02\rangle$, $\langle 12\rangle$, $\langle 21\rangle$, $\langle 10\rangle$, $\langle 20\rangle$. All other ground states coexist at zero.

In the first order ($E_1 = 3$),

$$n_1^G(0) = l_{\alpha^2 o}(G) + l_{o\alpha^2}(G).$$

Although condition 2 is not satisfied, n_1^G has the form (4.12) with C being the empty sequence ($l_{\alpha^2 o}$ is the proper extension of $l_{\alpha o}$ since the sequence $o\alpha o$ is not allowed). There is a new phase $\langle \alpha^2 o\rangle$ between $\langle \alpha^2\rangle$ and $\langle o\rangle$. Phases coexisting at the boundary $\langle \alpha^2\rangle - \langle \alpha^2 o\rangle$ are defined by the condition: $l_{oo}(G) = 0$, so they are of the form (4.1) with $A = \alpha^2$, $B = o$, $u = 1$. Condition 2 is satisfied, and so we can start the induction. In the inductive step, we consider a subset of ground states (4.1) with both A and B in the form

$$\alpha^{2k_1} o \alpha^{2k_2} o \ldots o \alpha^{2k_m} o. \tag{5.1}$$

The common core has a similar form, with α at both ends. By laborious inspection one shows that in the suitable order N defined by E_N, the coefficient a_N of l_{oCo} is

$$E_N = 5\sum_{i=1}^{m} k_i + m \qquad a_N = -\frac{1}{(2m)!}\, k_1 k_2 \ldots k_m.$$

By theorem 2, the new phase $\langle A^{u+1}B\rangle$ appears and phases coexisting on borderlines $\langle A\rangle$-$\langle A^{u+1}B\rangle$ and $\langle A^{u+1}B\rangle$-$\langle A^u B\rangle$ are again of the form (4.1). The phase diagram in the vicinity of the borderline $\langle \alpha^2\rangle$-$\langle \alpha^2 o\rangle$ has the following form. In order $E_{N_1} = 11$ the phase $\langle \alpha^4 o\rangle$ appears between $\langle \alpha^2\rangle$ and $\langle \alpha^2 o\rangle$. In order $E_{N_2} = 16$ the phase $\langle \alpha^6 o\rangle$ shows up between $\langle \alpha^2\rangle$ and $\langle \alpha^4 o\rangle$, while in order $E_{N_3} = 12$ the phase $\langle \alpha^2 o \alpha^4 o\rangle$ appears between $\langle \alpha^2 o\rangle$ and $\langle \alpha^4 o\rangle$. In suitable higher orders, $\langle \alpha^8 o\rangle$ appears between $\langle \alpha^2\rangle$ and $\langle \alpha^6 o\rangle$, $\langle \alpha^4 o \alpha^6 o\rangle$ between $\langle \alpha^4 o\rangle$ and $\langle \alpha^6 o\rangle$, $\langle \alpha^2 o(\alpha^4 o)^2\rangle$ between $\langle \alpha^4 o\rangle$ and $\langle (\alpha^2 o)^2 \alpha^4 o\rangle$ and $\langle (\alpha^2 o)^2 \alpha^4 o\rangle$ between $\langle \alpha^2 o\rangle$ and $\langle \alpha^2 o \alpha^4 o\rangle$ (cf figure 1). This process then continues

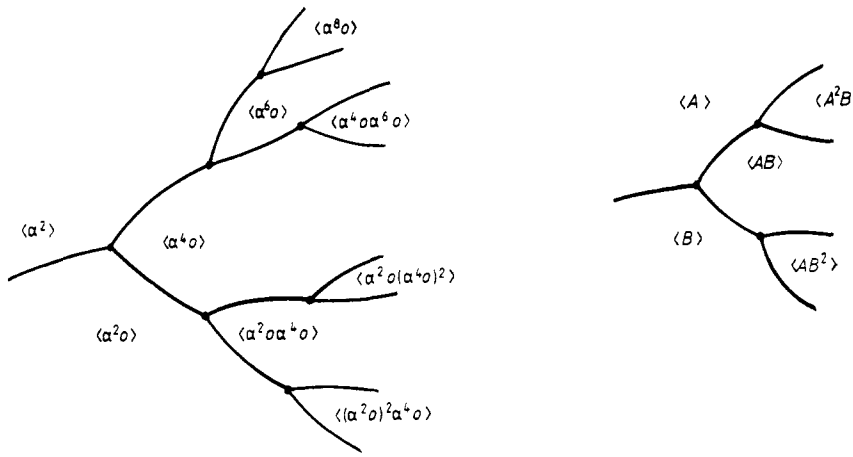

Figure 1. The schematic representation of the devil's staircase appearing at the locus of the boundary $\langle \alpha^2\rangle$-$\langle \alpha^2 o\rangle$. Any boundary between phases $\langle A\rangle$ and $\langle B\rangle$ bifurcates in some higher order in the manner shown at the right part of the diagram. Here $\langle \alpha^2\rangle$ is exemplified by the periodic sequence of spin values: 010101..., and $\langle o\rangle$ by 012012... (cf text for notation).

with no borderline being stable. Hence we obtain the phase diagram resembling in some sense the devil's staircase. A similar feature has been found by Uimin [8] for the ANNNI model in a non-zero magnetic field†.

The analysis of the boundary $\langle\alpha^2 o\rangle$-$\langle o\rangle$ shows the phase $\langle\alpha^2 o^2\rangle$ which appears in order $E_2 = 6$. No other phases emerge in this part of the phase diagram in any higher order.

6. Conclusions

In this paper we have discussed phase diagrams for a class of layered systems defined by condition 1. Phase diagrams were studied by means of an inductive argument based on the LTE technique. We have shown that in each inductive step, the search for new phases reduces to determining the contribution from a restricted set of excitations. This set is given in general terms, hence the layered structure forms the basis for the emergence of an infinite set of phases. The final form of the phase diagram is determined by specific energetical properties of the system considered. The idea behind the analysis is in some ways a generalisation of the SF method, with different technical details.

The generalisations of our method can be outlined as follows. First, one can consider a wider class of layered systems, either by changing the basic definition (the requirement that ground states are constant in a layer) or by modifying condition 1. In fact theorem 2, the main element of the inductive step, requires assumptions weaker than condition 1. Second, at the input of the inductive step one can allow sequences constructed of more than just two basic sequences. Finally, systems with many parameter perturbations can be considered. We remark that there is no suitable example so far for these generalisations.

The open problem is the interpretation of the phase diagram with an infinite number of phases, in view of the potential divergence of the LTE series. For a class of systems where the set of ground states is finite, these series are shown to be asymptotic [5] at least for some values of the perturbation parameters. We do not know about corresponding results for the case of an infinite set of ground states.

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Appendix 1. The phase diagram for a set of affine functional

Let $\Gamma = \{\rho_i\}$ be a countable set of affine functionals: $\rho_i = (h_i, a_i)$ with $h_i : \mathbb{R} \to \mathbb{R}$ linear and $a_i \in \mathbb{R}$. Assume that $W = \operatorname{conv}\Gamma$ is bounded in $\mathbb{R}^2$. We say that ρ dominates at $x \in \mathbb{R}$ if $\rho(x) \geq \rho'(x)$ for all ρ' in Γ. The phase diagram for Γ is the separation of $\mathbb{R}$ into regions of a single functional dominance. It is described by extremal properties of the set of maxima of W in the following way.

† This reference has been drawn to my attention by one of the referees of this paper.

Lemma. Suppose that max W has a finite number of extremal points.

(i) If F is an extremal edge spanned by ρ_1 and ρ_2, then there exists $x(F) \in \mathbb{R}$ such that for any $\rho' \notin F$

$$\rho(x(F)) = \text{constant} > \rho'(x(F)).$$

(ii) If ρ_0 is an extremal point belonging to F_1 and F_2 (extremal edges) then the region of its dominance is non-empty and is contained between $x(F_1)$ and $x(F_2)$.

(iii) If $\rho_0 \in F$ (an extremal edge) but is not an extremal point then it dominates only at $x(F)$.

(iv) If $\rho_0 \notin \max W$ then it does not appear in the phase diagram.

Proof. Recall that for a convex set W, a line P supporting W at ρ_0 is defined as $P = \{\rho: \rho(x(P)) = \rho_0(x(P))\}$ where $x(P) \in \mathbb{R}$ is such that W lies entirely on one side of P. If ρ_0 is in max W, then the last condition has the form:

$$\rho_0(x(P)) \geqslant \rho(x(P)) \qquad \rho \in W.$$

(i) F is an extremal edge and ρ_1, ρ_2 are its endpoints. Then F is contained in the unique supporting line P passing through ρ_1 and ρ_2. Define $x(F) = x(P)$.

(ii) ρ_0 is an extremal point. Then there exists at least one supporting line intersecting W only at ρ_0 (using the Krein-Millman theorem). Obviously ρ_0 dominates at $x(P)$. Let $\rho_0 \in F_1 \cap F_2$. Without loss of generality, we may assume that $x(F_1) < x(F_2)$. If $x \notin [x(F_1), x(F_2)]$, then W lies on both sides of $P(x) = \{\rho: \rho_0(x) = \rho(x)\}$. If $x \in [x(F_1), x(F_2)]$ and ρ_0 does not dominate at x, then either F_1 or F_2 is not the extremal edge.

(iii) Let $\rho_0 = \lambda\rho_1 + (1-\lambda)\rho_2$ with ρ_1, ρ_2 being endpoints of F. Obviously ρ_0 dominates at $x(F)$. If $x \neq x(F)$, then $\rho_1(x) \neq \rho_2(x)$ and ρ_0 is dominated either by ρ_1 or ρ_2.

(iv) Let $\rho_0 = (h_0, a_0)$, and $\rho_0' = (h_0, a)$ with a such that $\rho' \in \max W$. Obviously $a > a_0$, hence ρ_0' dominates ρ_0 everywhere. But $\rho_0' = \lambda\rho_1 + (1-\lambda)\rho_2$ for some $\lambda \in [0_1 1]$. Hence for any x, ρ_0 is dominated either by ρ_1 or ρ_2.

Appendix 2. The proof of theorem 2

Let ρ_G be the affine functional (4.12). Without loss of generality we assume that $e_{\langle A\rangle} > e_{\langle A^\mu B\rangle}$.

Case (a): $a_N > 0$. We claim that max conv$\{\rho_G\}$ has only two extremal points corresponding to $\langle A\rangle$ ($\rho_{\langle A\rangle} = (-e_{\langle A\rangle}, |A|^{-1}a_N)$) and $\langle A^\mu B\rangle$ ($\rho_{\langle A^\mu B\rangle} = -e_{\langle A^\mu B\rangle}, 0)$). Let F be an edge spanned by $\rho_{\langle A\rangle}$ and $\rho_{\langle A^\mu B\rangle}$. Then

$$x(F) = \frac{a_N}{|A|} \frac{1}{e_{\langle A\rangle} - e_{\langle A^\mu B\rangle}}.$$

Suppose that G is such that ρ_G lies in or above F. Then

$$a_N l_{\mu C\bar{\mu}}(G) - e_G x(F) \geqslant e_{\langle A^\mu B\rangle} x(F).$$

By (4.2), e_G is a linear function of $l_{A^{\mu+1}}$. Substituting for e_G and $x(F)$ and performing simple transformations, one obtains

$$l_{\mu C\bar{\mu}}(G) - l_{A^{\mu+1}}(G) \geqslant 0.$$

Since μC has a common proper extension with $A^{\mu+1}$, the above condition is equivalent to

$$0 \geqslant l_{\mu C}(G) - l_{\mu C\bar{\mu}}(G) = l_{\mu C\bar{\nu}}(G).$$

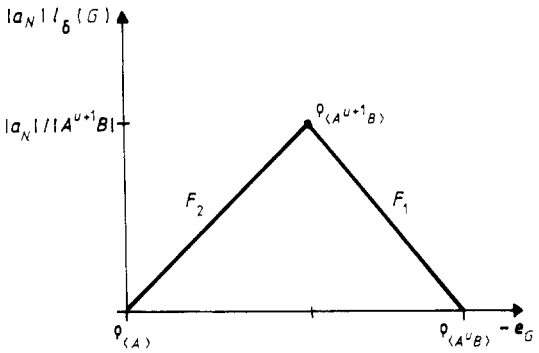

Figure 2. The set max W in the case (b) of theorem 2. Extremal points correspond to phases $\langle A\rangle$, $\langle A^{u+1}B\rangle$ and $\langle A^uB\rangle$. The set W is the triangle spanned by these extremal points. Max W is the bold line.

But $\mu C\bar{\nu}$ has a common proper extension with $A^{u+1}B$, thus $l_{A^{u+1}B}(G) \leqslant 0$. Hence G can at most lie in F_1 and in this case $l_{A^{u+1}B}(G) = 0$. The only ground states satisfying this condition are $\langle A\rangle$ and $\langle A^uB\rangle$. By the lemma of appendix 1, no other ground states appear in the phase diagram.

Case (b): $a_N < 0$. Using structural relations (2.3), one can rewrite (4.12) to the form (modulo constant term and term proportional to e_G):

$$n_N^G(x) = \tfrac{1}{2}|a_N(x)|(l_\mu C_{\bar{\nu}}(G) + l_{\nu C\bar{\mu}}(G)) \equiv |a_N(x)| l_\delta(G).$$

We claim that max conv$\{\rho_G\}$ appears in this case as in figure 2. To see this, let us consider the edge F_1 spanned by affine functional corresponding to $\langle A^uB\rangle$ and $\langle A^{u+1}B\rangle$. Then

$$x(F_1) = -\frac{|a_N|}{|A^{u+1}B|}(e_{\langle A^{u+1}B\rangle} - e_{\langle A^uB\rangle})^{-1}. \tag{A2.1}$$

Suppose that ρ_G lies in or above F_1, then $e_{\langle A^{u+1}B\rangle} > e_G > e_{\langle A^uB\rangle}$ and

$$|a_N| l_\delta(G) + e_G x(F_1) \geqslant e_{\langle A^uB\rangle} x(F_1).$$

By (4.2), e_G is a linear function of l_{A^uB}. Substituting for e_G and $x(F_1)$, after transformations one obtains:

$$0 \geqslant l_{A^uB}(G) - l_\delta(G) = \tfrac{1}{2}(l_{A^uB}(G) - l_{\mu C\bar{\nu}}(G) + l_{BA^u}(G) - l_{\nu C\bar{\mu}}(G)).$$

($l_{A^uB}(G) = l_{BA^u}(G)$ because of periodicity.) Since A^uB has a common proper extension with $C\bar{\nu}$ this condition is equivalent to: $0 \geqslant l_\nu c_{\bar{\nu}}(G)$. Hence ρ_G can at most lie in F_1, and in this case $l_{\nu C\bar{\nu}}(G) = 0$, so G has the form (4.1). The proof for the edge F_2 goes along similar lines. The application of the lemma of appendix 1 shows that the phase $\langle A^{u+1}B\rangle$ appears. Other statements of theorem 2 also hold, with the formula for the width of the region occupied by $\langle A^{u+1}B\rangle$ obtained by subtracting (A2.1) from the similar expression for $x(F_2)$(where A^uB is replaced by A).

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